

Playing a quantum game on polarization vortices

A. R. C. Pinheiro¹, C. E. R. Souza¹, D. P. Caetano², J. A. O. Huguenin³, A. G. M. Schmidt³, and A. Z. Khoury¹

¹ Instituto de Física, Universidade Federal Fluminense, 24210-346 Niterói - RJ, Brasil.

² Escola de Engenharia Industrial Metalúrgica, Universidade Federal Fluminense, 27255-125 Volta Redonda - RJ, Brazil and

³ Instituto de Ciências Exatas, Universidade Federal Fluminense, 27213-415 Volta Redonda - RJ, Brazil

The quantum mechanical approach to the well known prisoners dilemma, one of the basic examples to illustrate the concepts of Game Theory, is implemented with a classical optical resource, nonquantum entanglement between spin and orbital degrees of freedom of laser modes. The concept of entanglement is crucial in the quantum version of the game, which brings novel features with a richer universe of strategies. As we show, this richness can be achieved in a quite unexpected context, namely that of paraxial spin-orbit modes in classical optics.

PACS numbers: 03.67.Ac, 42.50.Ex

Numerous quantum information protocols rely on entanglement, a property usually attributed to composite quantum systems that cannot be described by a tensor product state vector. Quantum cryptography and teleportation are popular examples of protocols relying on quantum entanglement. While this property is considerably sensitive to local measurements performed in each party of the composite system, it is unaffected by local unitary operations. Teleportation protocols, for example, are achieved by classical communication followed by local unitary operations and measurements. This framework is where Quantum Mechanics meets an important area of applied Mathematics, the Game theory, a powerful tool for decision making [1, 2]. Here, two or more agents (players) take their decisions by acting on a quantum system with unitary operations. These decisions or conflict situations can be as simple as tossing a coin[1] or rather involved like the so-called minority game [3], where one models a competition among several players for a limited resource. In this sense games can be co-operative or non-cooperative like the prisoners dilemma, and with complete (incomplete) information where one player knows (does not know) all strategies his opponent can choose. Quantum versions of this game was realized experimentally both with Nuclear Magnetic Resonance [4] and entangled photon pairs [5].

Although frequently attributed to quantum systems, entanglement has been recently identified in classical optics as the coherent superposition of paraxial modes with orthogonal spatial profiles and orthogonal polarizations. We shall refer to such superpositions as *spin-orbit modes*. These modes were used in our group as a classical optical resource to investigate the topological phase acquired by an entangled state following a cyclic evolution under local unitary operations [6] and to demonstrate alignment free BB84 quantum cryptography ref.[7]. Also, a spin-orbit Bell inequality has been investigated both in the quantum [8] and classical [9] domains. An important tool for spin-orbit coupling was used in ref.[10] to exchange quantum information between these two degrees of freedom. In a recent work [11] the term *nonquantum*

entanglement has been coined to designate the spin-orbit inseparability of paraxial modes in connection with the Mueller matrices employed in polarization optics. In this work we demonstrate how this kind of nonquantum entanglement can be used to evaluate the performance of quantum strategies in a classical example of game theory, the prisoners dilemma.

A laser beam propagating along the z direction is usually described by its polarization unit vector \hat{e} and its spatial mode $\psi(\mathbf{r})$. The spatial modes in rectangular coordinates are Hermite-Gaussian (HG) solutions of the paraxial wave equation described in many text books [12]. The subspace of first order spatial modes has a qubit structure similar to the polarization mode space, where HG modes along different orientations play the role of linear polarizations and the Laguerre-Gaussian (LG) modes are equivalent to circular polarization [13]. By combining the two mode spaces, a general spin-orbit mode can be written as

$$\begin{aligned} \Psi(\mathbf{r}) = & \alpha \psi_h(\mathbf{r}) \hat{e}_H + \beta \psi_v(\mathbf{r}) \hat{e}_H \\ & + \gamma \psi_h(\mathbf{r}) \hat{e}_V + \delta \psi_v(\mathbf{r}) \hat{e}_V, \end{aligned} \quad (1)$$

where $\hat{e}_{H(V)}$ are linear polarization unit vectors along the horizontal (vertical) directions, and $\psi_{h(v)}(\mathbf{r})$ are HG spatial modes along these same directions. In analogy to the usual entanglement measure used for bipartite quantum states [14], we can define the spin-orbit mode concurrence, which for eq.(1) is $C = |\alpha\delta - \beta\gamma|$. We shall refer to the spin-orbit modes with maximal concurrence $C = 1$ as maximally entangled modes. In these modes neither the polarization nor the spatial profile is well defined, in fact they correspond to polarization vortices which have been extensively studied due to their potential applications to high resolution microscopy [15, 16]. In this work we demonstrate another appealing feature in the unexpected context of a quantum game. An example of such entangled spin-orbit mode is

$$\Psi_0(\mathbf{r}) = \frac{\psi_h(\mathbf{r}) \hat{e}_H + i \psi_v(\mathbf{r}) \hat{e}_V}{\sqrt{2}}. \quad (2)$$

We use this mode to implement a classical optical version of the well known prisoners dilemma, in which two players, Alice and Bob, accused of a felony, have to decide whether they cooperate (C) or defeat (D) each other. In classical game theory, each agent decision is represented by one bit of information with possible states C and D . Depending on their decision, a reduced penalty may be applied to each one of them. The penalty reduction is the payoff each player gets from their combined decisions. The penalty reductions for both players are shown in table I for all possible strategies adopted by Alice (rows) and Bob (columns). From the table we see that both

(R_A, R_B)	C	D
C	(3,3)	(0,5)
D	(5,0)	(1,1)

TABLE I: Penalty reduction table

players are tempted to choose D , although their added reduction would be maximized by CC . Here comes the dilemma, the players are isolated without the permission to negotiate. Each player is left to his own and has to decide whether to defeat or cooperate with the other, a bad choice may cost his freedom. For a cooperative game, players would choose strategies which maximize both payoffs, i.e., they would search for Pareto optimal strategies; on the other hand, since prisoners dilemma is a non-cooperative game, each player will try to maximize solely his own payoff, i.e., the intelligent choice is the Nash equilibrium DD . At this point, the concepts of Game Theory are due. Suppose this situation is repeated many times and the players adopt probabilistic strategies, that is, they randomly choose between C and D with prestablished probabilities. Then, the payoff function of each player is given by the average penalty reduction obtained:

$$\mathbb{S}_j = \sum_{m,n=C,D} p(m,n) R_j(m,n), \quad (3)$$

where $j = A, B$ and $p(m,n) = p_A(m)p_B(n)$ is the joint probability that Alice chooses m and Bob chooses n . In the classical approach to the problem, one shows that the payoff as a function of $p_A(m)$ and $p_B(n)$ has an absolute minimum at $p_A(D) = p_B(D) = 1$, that is the best the players can do is to defeat, as a consequence of the severe cost brought by a possible betrayal.

In ref.[17] Eisert proposes an ingenious alternative to the classical approach, employing the concept of quantum entanglement. Briefly, in this approach the prisoners share a pair of entangled qubits and rather than making a definite C or D statement, they are allowed to perform single qubit unitary operations (strategies), each one on the qubit in his possession. The entangled

two-qubit state is prepared by a nonlocal operation

$$U = \begin{bmatrix} 1 & 0 & 0 & i \\ 0 & 1 & i & 0 \\ 0 & i & 1 & 0 \\ i & 0 & 0 & 1 \end{bmatrix} \quad (4)$$

acting on an initial state $|CC\rangle$, so that $U|CC\rangle = (|CC\rangle + i|DD\rangle)/\sqrt{2}$. After each player has applied his own strategy U_j ($j = A, B$), the qubits are nonlocally operated with U^\dagger and separately measured. The payoff function (3) is evaluated with the probabilities $p(m,n) = \langle mn|U^\dagger(U_A \otimes U_B)U|CC\rangle$ ($m,n = C,D$) associated with the two possible outcomes (C or D) in each qubit. Therefore, the strategy space is much larger in this quantum approach, it corresponds to the space of $SU(2) \otimes SU(2)$ matrices, apart from irrelevant global phases. This quantum version has been implemented experimentally with quantum correlated photon pairs generated by parametric down conversion [18].

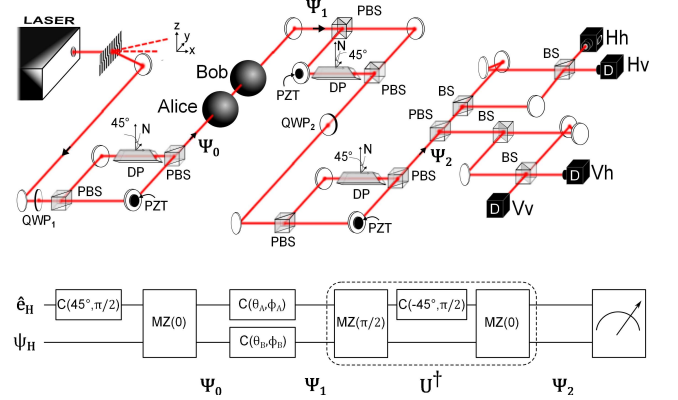


FIG. 1: Experimental setup.

We now demonstrate the implementation of the quantum game strategies with the nonquantum entanglement provided by the spin-orbit classical mode (2). In the game language, we shall make the identification $H \equiv C$ and $V \equiv D$. The experimental setup is shown in fig.(1). The TEM₀₀ output of a He-Ne laser is diffracted by a hologram that generates an HG mode on the first diffraction order, so that the initial spin-orbit mode is $\psi_h(\mathbf{r})\hat{e}_H$. This mode is first sent to a quarter wave plate (QWP) rotated at 45° , which makes the transformation $\hat{e}_H \rightarrow (\hat{e}_H + i\hat{e}_V)/\sqrt{2}$, and then to a Mach-Zehnder (MZ) interferometer with input and output polarizing beam splitters (PBS). A Dove prism (DP) is inserted in one arm of the MZ interferometer, and the relative phase ϕ between the two arms is controlled by a piezoelectric transducer (PZT). The DP oriented at 45° makes the transformation $\psi_h \rightarrow \psi_v$, so that this MZ interferometer coherently superposes modes $\psi_v(\mathbf{r})\hat{e}_V$ and $\psi_h(\mathbf{r})\hat{e}_H$ at its output with an adjustable phase ϕ . In the basis

$\{\psi_h(\mathbf{r})\hat{e}_H, \psi_v(\mathbf{r})\hat{e}_H, \psi_h(\mathbf{r})\hat{e}_V, \psi_v(\mathbf{r})\hat{e}_V\}$, the matrix representation of the MZ transformation is:

$$MZ(\phi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -e^{i\phi} \\ 0 & 0 & e^{i\phi} & 0 \end{bmatrix}, \quad (5)$$

so that after passing through the QWP and the balanced interferometer ($\phi = 0$), the beam is prepared in mode (2). This mode is the object on which the players will implement their strategies.

In order to continue our experimental description, it is essential to define the mode converter operators which will be used to implement the players strategies and U^\dagger . We will be dealing with either polarization (wave plates) or spatial (DP and cylindrical lenses) mode converters, that is, elements acting on one degree of freedom only. We shall inform rotation angles in degrees and phase retardations in radians for immediate identification of the physical meaning all over the experimental description. When oriented along horizontal and vertical directions, they introduce a retardation phase ϕ between H and V modes. A mode converter rotated by the angle θ is described by the $SU(2)$ matrix:

$$C(\theta, \phi) = \begin{bmatrix} \cos \frac{\phi}{2} + i \sin \frac{\phi}{2} \cos 2\theta & i \sin \frac{\phi}{2} \sin 2\theta \\ i \sin \frac{\phi}{2} \sin 2\theta & \cos \frac{\phi}{2} - i \sin \frac{\phi}{2} \cos 2\theta \end{bmatrix}. \quad (6)$$

For example, quarter wave plates correspond to $\phi = \pi/2$ and half wave plates (HWP) to $\phi = \pi$. Spatial mode converters can be made with cylindrical lenses [19] for variable retardation ϕ , or the DP for $\phi = \pi$. Now, Alice is equipped with polarization elements and realizes strategies of the kind $C(\theta_A, \phi_A)$, whereas Bob is equipped with DPs and cylindrical lenses, and his strategies are $C(\theta_B, \phi_B)$. After the players have made their choices, the spin-orbit mode of the laser beam is:

$$\Psi_1(\mathbf{r}) = [C(\theta_A, \phi_A) \otimes C(\theta_B, \phi_B)] \Psi_0(\mathbf{r}). \quad (7)$$

The mode converters are also used together with $MZ(\phi)$ to implement U^\dagger . Indeed, one can easily show that:

$$U^\dagger = MZ(0) [C(-45^\circ, \pi/2) \otimes I] MZ(\pi/2), \quad (8)$$

which physically means that, after passing through the players strategies, the beam is sent through a MZ interferometer with a $\pi/2$ phase shift, a quarter wave plate (QWP) rotated at -45° , and another MZ interferometer with balanced arms ($\phi = 0$), where it gets transformed to mode

$$\Psi_2(\mathbf{r}) = U^\dagger \Psi_1(\mathbf{r}) = \sum_{m,n} c_{mn} \psi_m(\mathbf{r}) \hat{e}_n, \quad (9)$$

where $m = h, v$ and $n = H, V$.

After this transformation, we arrive at the measurement stage where the probabilities $p(m, n)$ used in the payoff function eq.(3) are obtained. In our setup, these probabilities are given by the projected intensities:

$$p(m, n) = |c_{mn}|^2 = \left| \int d^2\mathbf{r} \psi_m^*(\mathbf{r}) [\hat{e}_n^* \cdot \Psi_2(\mathbf{r})] \right|^2, \quad (10)$$

which, in fact, correspond to photodetection probabilities when the beam is attenuated down to the single photon regime. In order to measure the projected intensities, the $\Psi_2(\mathbf{r})$ mode is first sent to a PBS where polarization projection is performed. Then, each PBS output is sent to a balanced Mach-Zehnder interferometer with an additional mirror in one arm (MZIM), where spatial mode projection occurs [20]. The projected intensities are measured at the four outputs of the two MZIMs either with a CCD camera or with photodetectors. The payoff function is evaluated with the intensities measured with four photodetectors. The background noise is subtracted and the partial intensities are then normalized to the total intensity so that $p(m, n) \equiv I_{m,n}/I_{TOT}$.

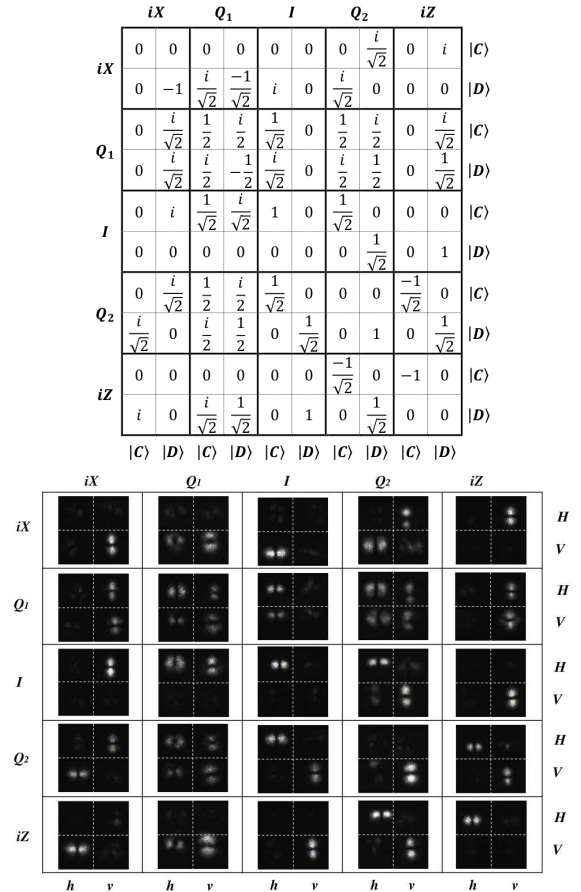


FIG. 2: a) Coefficients table of the final mode in the computational basis. b) Images of the corresponding output ports.

For each player, we implemented five different strate-

gies: $iX \equiv C(45^\circ, \pi)$, $Q_1 \equiv C(45^\circ, \pi/2)$, $I \equiv C(\theta, 0)$, $Q_2 \equiv C(0, \pi/2)$, and $iZ \equiv C(0, \pi)$. In this notation X and Z are the usual Pauli matrices. Strategies I and iX are equivalent to the classical ones where the players can only cooperate or defeat, while the other strategies are intrinsically quantum mechanical since they involve rotations and phase retardations not available in the classical scenario. A table with the coefficients c_{mn} resulting from these strategies is shown in fig.(2a). Also, the images obtained with the CCD camera are displayed in the same table format in fig.(2b). As expected, only those output ports corresponding to nonzero coefficients are illuminated. In this sense, the qualitative agreement between the two tables is clear. We have also evaluated Alice's payoff as a function of the strategy parameters (θ_A, ϕ_A) and (θ_B, ϕ_B) in the domain $(\theta = 0, 0 \leq \phi \leq \pi)$ and $(\theta = 45^\circ, 0 \leq \phi \leq \pi)$. The analytical result is shown in fig.(3) together with the points corresponding to all possible combinations of the experimental strategies. The experimental values were obtained from the intensity measurements, where the relative intensities play the role of measurement probabilities. We observe, from both theoretical and experimental results, that the quantum move $U_A = U_B = iZ$ proposed originally in [17] dominates all classical ones and it is also a Nash equilibrium with a better outcome than $U_A = U_B = iX$ since no player can improve his respective payoff by changing unilaterally his strategy. Note also that if Bob could choose only between classical $U_B = I$ or $U_B = iX$, and Alice keep $U_A = iZ$ then it would be better for him to cooperate since his payoff would be increased.

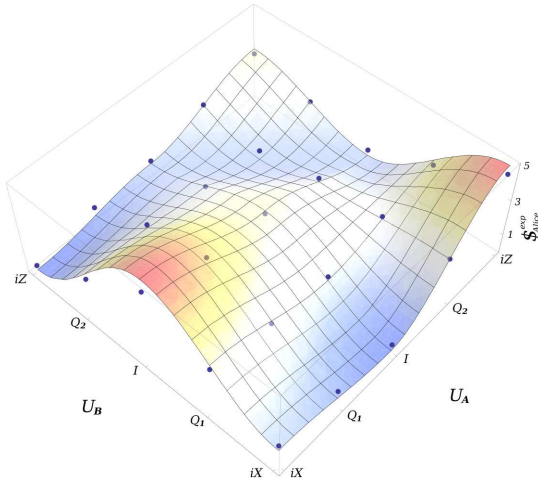


FIG. 3: Alice payoff as a function of the strategies parameters (θ_A, ϕ_A) and (θ_B, ϕ_B) . Dots correspond to the experimental values obtained with the intensity measurements.

In conclusion, we have used the concept of nonquantum entanglement to implement a Game Theory task in the context of the well known prisoners dilemma. The advantages offered by the quantum mechanical approach

could be realized in the classical optics framework. Non-separable spin-orbit modes corresponding to polarization vortices were used. This implementation opens promising perspectives regarding potential applications of non-quantum entanglement to the investigation of quantum information protocols.

Acknowledgments

Funding was provided by Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES), Fundação de Amparo à Pesquisa do Estado do Rio de Janeiro (FAPERJ-BR), and Instituto Nacional de Ciência e Tecnologia de Informação Quântica (INCT-CNPq).

-
- [1] D. A. Meyer, Phys. Rev. Lett. **82**, 1052 (1999).
 - [2] H. Guo, J. Zhang, and G. J. Koehler, Decision Support Systems **46**, 318 (2008).
 - [3] D. Challet, Y.-C. Zhang, Physica A **246**, 407 (1997).
 - [4] Jiangfeng Du, Hui Li, Xiaodong Xu, Mingjun Shi, Jihui Wu, Xianyi Zhou, and Rongdian Han, Phys. Rev. Lett. **88**, 137902 (2002).
 - [5] C. Schmid, A. P. Flitney, W. Wicorek, N. Kiesel, H. Weinfurter, and L. C. L. Hollenberg, New J. Phys. **12**, 063031 (2010).
 - [6] C. E. R. Souza, J. A. O. Huguenin, P. Milman, and A. Z. Khoury, Phys. Rev. Lett. **99**, 160401 (2007).
 - [7] C. E. R. Souza, C. V. S. Borges, A. Z. Khoury, J. A. O. Huguenin, L. Aolita, and S. P. Walborn, Phys. Rev. A **77**, 032345 (2008).
 - [8] L. Chen and W. She, J. Opt. Soc. Am. B, **27**, A7 (2010).
 - [9] C. V. S. Borges, M. Hor-Meyll, J. A. O. Huguenin, and A. Z. Khoury, Phys. Rev. A **82**, 033833 (2010).
 - [10] E. Nagali, F. Sciarrino, F. De Martini, L. Marrucci, B. Piccirillo, E. Karimi, and E. Santamato. Phys. Rev. Lett. **103**, 013601 (2009).
 - [11] B. N. Simon, S. Simon, F. Gori, M. Santarsiero, R. Borghi, N. Mukunda, and R. Simon, Phys. Rev. Lett. **104**, 023901 (2010).
 - [12] A. Yariv, Quantum Electronics, John Wiley & Sons (1989).
 - [13] M. J. Padgett and J. Courtial, Opt. Lett. **24**, 430 (1999).
 - [14] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).
 - [15] A. A. Ishaaya, L. T. Vuong, T. D. Grow and A. L. Gaeta, Opt. Lett. **33**, 13 (2008);
 - [16] V. V. G. K. Inavalli and N. K. Viswanathan, Opt. Comm. **283**, 861 (2010).
 - [17] J. Eisert, M. Wilkens, and M. Lewenstein, Phys. Rev. Lett. **83**, 3077 (1999).
 - [18] R. Prevedel, A. Stefanov, P. Walther, and A. Zeilinger, New J. Phys. **9**, 205 (2007).
 - [19] M.W. Beijersbergen, L. Allen, H.E.L.O van der Veen and J. P. Woerdman. Opt. Commun. **96**, 123 (1993).
 - [20] H. Sasada and M. Okamoto, Phys. Rev. A **68**, 012323 (2003).